Eur. Phys. J. A **28**, 259–263 (2006)
DOI 10.1140/epja/i2006-10061-5

THE EUROPEAN PHYSICAL JOURNAL A

Letter

Inelastic form factors to alpha-particle condensate states in ^{12}C and ^{16}O: What can we learn?

Y. Funaki[1,a], A. Tohsaki[2], H. Horiuchi[2], P. Schuck[3], and G. Röpke[4]

[1] The Institute of Physical and Chemical Research (RIKEN), Wako, Saitama 351-0198, Japan
[2] Research Center for Nuclear Physics (RCNP), Osaka University, Ibaraki, Osaka 567-0047, Japan
[3] Institut de Physique Nucléaire, 91406 Orsay Cedex, France
[4] Institut für Physik, Universität Rostock, D-18051 Rostock, Germany

Received: 11 April 2006 / Revised: 6 June 2006 /
Published online: 7 July 2006 –
Communicated by G. Orlandini

Abstract. In order to discuss the spatial extention of the 0_2^+-state of ^{12}C (Hoyle state), we analyze the inelastic form factor of electron scattering to the Hoyle state, which our 3α-condensate wave function reproduces very well like previous 3α RGM/GCM models. The analysis is made by varying the size of the Hoyle state artificially. As a result, we find that only the maximum value of the form factor sensitively depends on its size, while the positions of maximum and minimum are almost unchanged. This size dependence is found to come from a characteristic feature of the transition density from the ground state to the Hoyle state. We further show the theoretical predictions of the inelastic form factor to the 2_2^+-state of ^{12}C, which was recently observed above the Hoyle state, and of the inelastic form factor to the calculated 0_3^+-state of ^{16}O, which was conjectured to correspond to the 4α condensed state in previous theoretical work by the present authors.

PACS. 25.55.Ci Elastic and inelastic scattering – 03.75.Nt Other Bose-Einstein condensation phenomena – 21.60.Gx Cluster models – 27.20.+n $6 \le A \le 19$

It is well known that in lighter nuclei nuclear clusters, for instance α-particles, play a very important role. From the shell model point of view, one of the most mysterious states in light nuclei is the 0_2^+-state of 7.65 MeV in ^{12}C. It is known as the Hoyle state because of its importance in astrophysics [1]. On the other hand, this state has been explained with a microscopic three-alpha RGM wave function by Kamimura *et al.* [2] and Uegaki *et al.* [3]. More recently, we came to the same result with a two-parameter trial wave function which shows the aspect of the Hoyle state as a loosely bound condensate of α-particles with a volume 3 to 4 times the one of the ground state of ^{12}C [4, 5]. Since we have shown that our wave function has almost 100% overlap with the one of ref. [2], it comes as no surprise that we also reproduce the inelastic form factor $0_1^+ \to 0_2^+$ of ref. [2], which is in agreement with experiment. The significance of this finding will be strongly increased should there be an important dependence of the inelastic form factor on the size of the Hoyle state. One of the objectives of this short communication is precisely the study of this size dependence of the form factor. The other is that we will make predictions for two other inelastic form factors: $(0_1^+ \to 2_2^+)$ in ^{12}C and $0_1^+ \to$ predicted α-condensate state in ^{16}O.

The form factor is obtained by performing the Fourier transformation of the transition density as follows:

$$|F(q)|^2 = \frac{4\pi}{12^2}\Big|\int_0^\infty \rho_{J,0_1}^{(J)}(r) j_J(qr) r^2 \mathrm{d}r\Big|^2 \exp\Big(-\frac{1}{2}a_p^2 q^2\Big). \tag{1}$$

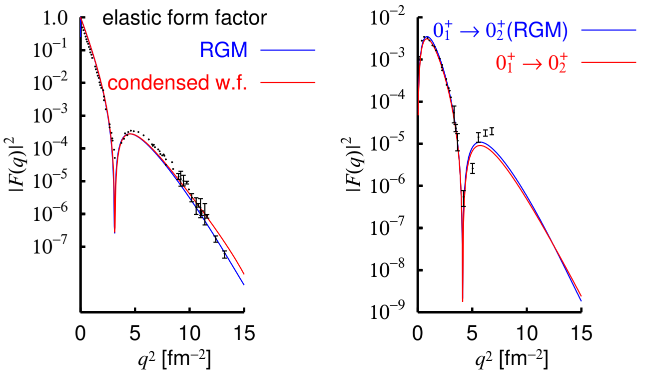
Here $a_p^2 = 0.43\,\mathrm{fm}^2$ is taken as the finite proton size, which is the same as adopted in ref. [2] and $j_J(qr)$ is the J-th order spherical Bessel function. The transition density $\rho_{J,0_1}^{(J)}(r)$ is given as

$$\rho_{J,0_1}^{(J)}(r) = \langle \Psi_{\lambda=k}^{JM}|\sum_{i=1}^{12}\delta(\boldsymbol{r}-\boldsymbol{r}_i)|\Psi_{\lambda=1}^{J=0}\rangle / Y_{JM}^*(\hat{\boldsymbol{r}}), \tag{2}$$

[a] e-mail: funaki@riken.jp

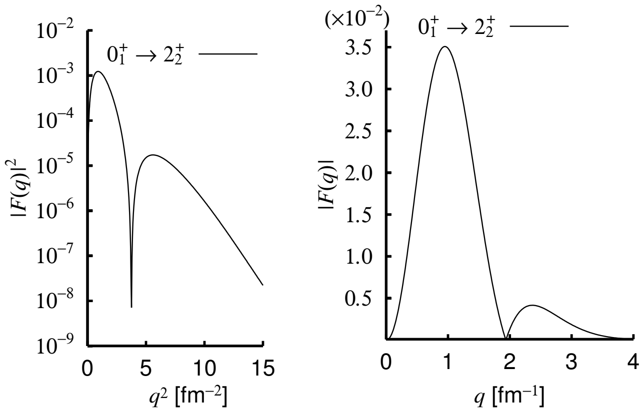
Table 1. Numerical values of elastic (upper row) and inelastic (lower row) form factors in ^{12}C.

q [fm^{-1}]	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
$\lvert F(q)_{0_1^+\to 0_1^+}\rvert(\times 10^{-1})$	9.4	7.7	5.5	3.3	1.6	0.56	0.030	0.15	0.16	0.11	0.065	0.033
$\lvert F(q)_{0_1^+\to 0_2^+}\rvert(\times 10^{-2})$	0.98	3.3	5.2	5.5	4.3	2.5	0.96	0.069	0.27	0.29	0.20	0.11



(a) (b)

Fig. 1. (Color online) (a) experimental values of the elastic form factor in ^{12}C are compared with our values obtained by solving the Hill-Wheeler equation, eq. (3), for $\Psi_{\lambda=1}^{J=0}$. The result given in ref. [2] using the resonating group method (RGM) is also shown. (b) Experimental values of the inelastic form factor in ^{12}C to the Hoyle state are compared with our values obtained by using the Hoyle state wave function $\Psi_{\lambda=2}^{J=0}$ and those given in ref. [2] (RGM). The experimental values are taken from ref. [6].



(a) (b)

Fig. 2. (a) Theoretical prediction of inelastic form factor to the 2_2^+-state of ^{12}C using the wave function of [7]. (b) The form factor shown in (a) is plotted as a function of q in linear scale for the ordinate.

where the ground and Hoyle states can be obtained by solving the following Hill-Wheeler equation:

$$\sum_{\boldsymbol{\beta}'} \langle \widehat{\Phi}_{3\alpha}^{N,J=0}(\boldsymbol{\beta})|(H-E)|\widehat{\Phi}_{3\alpha}^{N,J=0}(\boldsymbol{\beta}')\rangle f_{\lambda}^{J=0}(\boldsymbol{\beta}')=0, \quad (3)$$

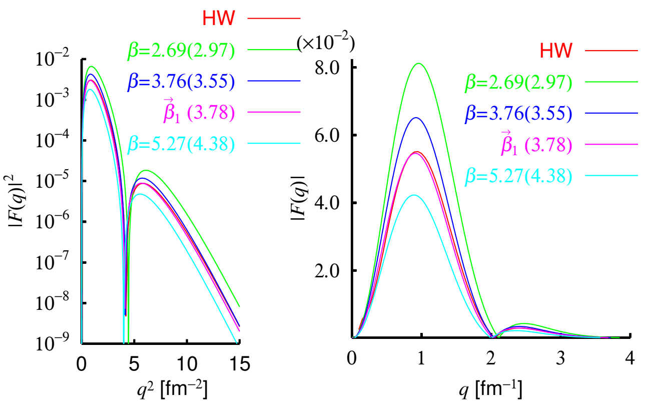
$$\Psi_{\lambda}^{J=0} = \sum_{\boldsymbol{\beta}} f_{\lambda}^{J=0}(\boldsymbol{\beta})\widehat{\Phi}_{3\alpha}^{N,J=0}(\boldsymbol{\beta}). \quad (4)$$

We here use the same notation for the alpha-condensate wave function, $\widehat{\Phi}_{3\alpha}^{N,J=0}(\boldsymbol{\beta})$ as was done in ref. [5]. The Hamiltonian H is the same as used in refs. [5,2]. The ground and Hoyle states correspond to the cases of $\lambda = 1$ and $\lambda = 2$ in eq. (4), respectively. Our results are shown in fig. 1 and we give our numerical values in table 1. Reflecting the fact that our wave functions of the ground state and the 0_2^+-state are almost equivalent to those given in ref. [2] using the resonating group method (RGM), our elastic and inelastic ($0_1^+ \to 0_2^+$) form factors almost completely agree with those given in ref. [2]. In fig. 2, we predict the inelastic form factor to the 2_2^+-state which is obtained in refs. [7,8] by using the 3α-condensate wave functions. This state was recently observed at 2.6 ± 0.3 MeV above the three-alpha threshold with the width of 1.0 ± 0.3 MeV [9], though the exsistence of this state has been suggested for a long time from the theoretical point of view [10]. Recently, the present authors carefully investigated this broad resonance state [7] using the ACCC method, which is known as a powerful method for the correct treatment of broad states beyond the bound-state approximation. As a result of the investigation, the 2_2^+-state was shown to be intimately related to the 0_2^+-state which is interpreted as the 3α Bose condensate state.

We now make a study of the sensitivity of the inelastic form factor with respect to some theoretical ingredients of our theory. A quantity of prime interest is the spatial extention of the Hoyle state which is predicted from our studies to have a volume 3 to 4 times as large as the one of the ground state of ^{12}C. We therefore repeated the calculation of the inelastic form factor in varying the size parameter of the Hoyle state. The calculation can be done as shown in ref. [5] by adopting as the Hoyle state the wave function, $\Psi_{\perp}(\boldsymbol{\beta}) \equiv \widehat{P}_{\perp}\widehat{\Phi}_{3\alpha}^{N,J=0}(\boldsymbol{\beta})$, where $\widehat{P}_{\perp}$ is defined as the projection operator onto the orthogonal space to the ground state:

$$\widehat{P}_{\perp} \equiv 1 - |\Psi_{\lambda=1}^{J=0}\rangle\langle\Psi_{\lambda=1}^{J=0}|. \quad (5)$$

Here the parameter $\boldsymbol{\beta}$ corresponds to the spatial extention of the alpha condensate. In fig. 3, the form factors from the ground state to the Hoyle state calculated at several $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)$ values are shown. Short-hand notations $\beta = \beta_x = \beta_y = \beta_z$ and $\boldsymbol{\beta}_1 := (\beta_x = \beta_y, \beta_z) =$



(a) (b)

Fig. 3. (Color online) (a) The inelastic form factors to the Hoyle state are plotted as a function of q^2. The wave function $\Psi_\perp(\boldsymbol{\beta})$ is adopted as the Hoyle state, the size of which is artificially changed by varying the values of the parameter $\boldsymbol{\beta}$. β is defined as $\beta \equiv \beta_x = \beta_y = \beta_z$ and $\boldsymbol{\beta}_1$ denotes $(\beta_x = \beta_y, \beta_z) = (5.27\,\text{fm}, 1.37\,\text{fm})$ [11]. The result using the wave functions of ground and Hoyle states which are obtained by solving the Hill-Wheeler equation is denoted by HW. (b) The form factors shown in (a) are replotted as a function of q in linear scale for the ordinate. The r.m.s. radii corresponding to $\Psi_\perp(\beta)$ are shown in parenthesis. Units of all numbers are in fm.

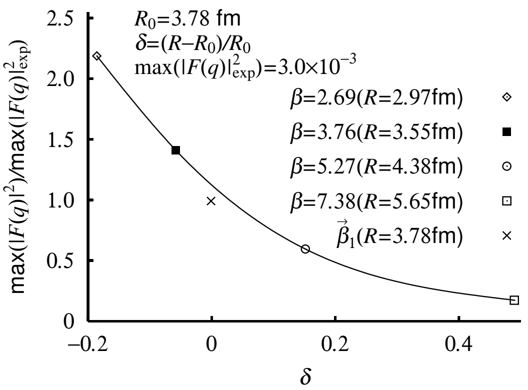


Fig. 4. The ratio of the value of maximum height, theory *versus* experiment, for the inelastic form factor, *i.e.* $\max|F(q)|^2/\max|F(q)|^2_{\text{exp}}$, is plotted as a function of δ, which is defined as $\delta = (R - R_0)/R_0$. R and R_0 are the r.m.s. radii corresponding to $\Psi_\perp(\beta = \beta_x = \beta_y = \beta_z)$ and $\Psi_\perp(\boldsymbol{\beta}_1)$, respectively. Here $R_0 = 3.78\,\text{fm}$. The unit of β is in fm.

$(5.27\,\text{fm}, 1.37\,\text{fm})$ [11] are used here and in the following. The corresponding r.m.s. radii are shown in parenthesis. We see that the overall structure of the form factor is not very much affected by artificially changing the radius of the Hoyle state, *i.e.*, for instance, the position of the minimum is only changed in very slight proportions. However, we can see that the amplitude decreases as the r.m.s. radius of the 0_2^+-state increases. In order to see how sensitively the height of the first maximum depends on the r.m.s. radius of the 0_2^+-state, we plot in fig. 4 the variation of this height with respect to the size of the Hoyle state and we see that this height changes *strongly* when the r.m.s. radius of the 0_2^+-state is changed. For example, it is seen from fig. 4 that an increase of 20% of the r.m.s. radius of the Hoyle state reduces the maximum of the form factor by 50%! It is therefore allowed to say that the measurement of the inelastic form factor of α-particle condensate states allows via our model wave function to *deduce the radius* of such a state. We should note that the wave function of the 0_2^+-state $\Psi^{J=0}_{\lambda=2}$ can be described rather well by $\Psi_\perp(\boldsymbol{\beta})$ as far as reasonable $\boldsymbol{\beta}$ values are adopted. The squared overlap between $\Psi^{J=0}_{\lambda=2}$ and $\Psi_\perp(\boldsymbol{\beta})$ amounts to 99.2% at $\boldsymbol{\beta} = \boldsymbol{\beta}_1$. Due to this almost complete equivalence between both wave functions, $\Psi^{J=0}_{\lambda=2}$ and $\Psi_\perp(\boldsymbol{\beta}_1)$, we understand that the corresponding inelastic form factors obtained by using both wave functions, *i.e.* denoted by HW and $\boldsymbol{\beta}_1$ in fig. 3, almost completely agree with one another. As for the other choices of $\boldsymbol{\beta}$, $\Psi_\perp(\boldsymbol{\beta})$ also has a reasonably large amount of squared overlap with $\Psi^{J=0}_{\lambda=2}$, *i.e.* 64.4%, 90.1%, and 81.8% at $\beta = 2.69\,\text{fm}$, $3.76\,\text{fm}$, and $5.27\,\text{fm}$, respectively. It should be emphasized that the fact that the wave function $\Psi_\perp(\boldsymbol{\beta})$, which is parametrized by β, is more or less a good approximation of the 0_2^+-state guarantees the validity of the above discussion of size dependence.

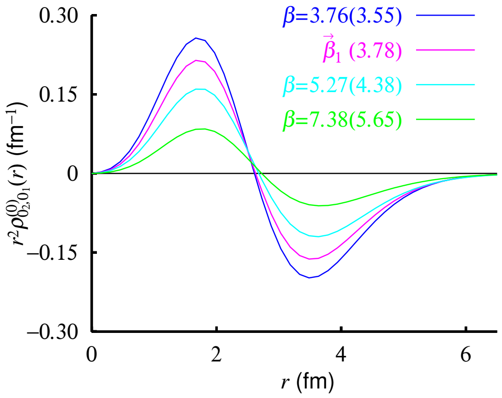


Fig. 5. (Color online) Transition densities defined in eq. (2) multiplied by r^2, *i.e.* $r^2\rho^{(0)}_{0_2 0_1}(r)$, corresponding to wave functions $\Psi_\perp(\boldsymbol{\beta})$ with different $\boldsymbol{\beta}(= \beta_x = \beta_y = \beta_z)$ values. $\boldsymbol{\beta}_1$ is given by $(\beta_x = \beta_y, \beta_z) = (5.27\,\text{fm}, 1.37\,\text{fm})$ [11]. The r.m.s. radii corresponding to $\Psi_\perp(\beta)$ are shown in parenthesis. Units of all numbers are in fm.

We can analyze the reason for these features of the form factor in the following simple way. In fig. 5 we show the transition density $r^2\rho^{(0)}_{0_2 0_1}(r)$ for different values of β. Due to the orthogonality between $\Psi^{J=0}_{\lambda=1}$ and $\widehat{P}_\perp \widehat{\Phi}^{N,J=0}_{3\alpha}(\boldsymbol{\beta})$ one has the relation $\int_0^\infty r^2\rho^{(0)}_{0_2 0_1}(r)\mathrm{d}r = 0$. We note that the position of the node of $r^2\rho^{(0)}_{0_2 0_1}(r)$ at $r \approx 2.5\,\text{fm}$ and the point where this transition density drops approximately to zero, *i.e.* at $r \approx 6.0\,\text{fm}$, do not depend on the various values of β. Also the feature of an approximate odd function around the nodal point holds for all β-values. It clearly can be concluded that one can approximately write

$$\rho^{(0)}_{0_2 0_1}(r) \approx f(\beta)\tilde{\rho}^{(0)}_{0_2 0_1}(r), \tag{6}$$

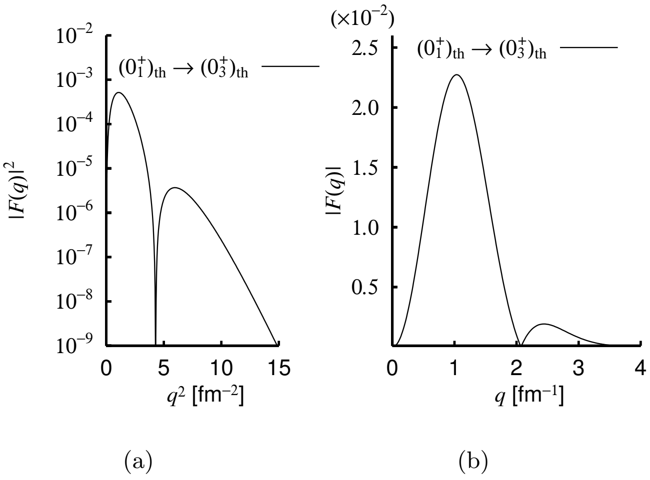


Fig. 6. (a) Inelastic form factor in ^{16}O to the 4α-condensate state as obtained from the wave function corresponding to the third eigen-energy state in ref. [4]. (b) The form factor shown in (a) is plotted as a function of q in linear scale for the ordinate.

where $f(\beta) \geq 0$ is independent of r and a decreasing function of β, whereas $\tilde{\rho}^{(0)}_{0_2 0_1}(r)$ is independent of β. This means that the form factor $F(q)$ just changes amplitude but not shape when the size of the Hoyle state is varied. This analysis is completely consistent with the features seen in fig. 3. Considering a simple situation of the wave function having a uniform density of spherical shape with the r.m.s. radius R, one can understand this size dependence as coming from the normalization factor of the wave function $R^{-3/2}$, which corresponds to $f(\beta)$ in eq. (6). In fact, the smooth curve shown in fig. 4 turns out to be close to $(R/R_0)^{-3} = (1+\delta)^{-3}$, which is just the squared ratio of the volumes of the states with the r.m.s. radii, R and R_0, respectively.

Concluding this point on the spatial extension of the Hoyle state we can say that the reproduction of the experimental data of the inelastic form factor is highly predictive and adds further confidence to the fact that the 0_2^+-state in ^{12}C has a very dilute structure with triple to quadruple volume of the ^{12}C ground state.

On these grounds we also want to make a prediction of the inelastic form factor to the α-condensate state in ^{16}O. In fig. 6 we show this form factor calculated with the α-particle condensate wave function for ^{16}O determined previously [4]. This latter state is actually the 3rd 0^+-state of our calculation, whose energy is at $E_{0_3^+} = 14.1$ MeV. We see that the inelastic form factor for ^{16}O resembles very much in its structure the one of ^{12}C. The positions of minimum and maximum are almost unchanged, while the height of the first maximum is relatively suppressed compared to the case of ^{12}C. A candidate of the 4α-condensate state may have been observed at 13.5 MeV with an alpha decay width of 0.8 MeV [12]. This new state is the 5th 0^+-state experimentally, corresponding to the 3rd 0^+-state of our calculation. An argument that in ^{16}O the α-condensate state is around 13.5 MeV could go as follows: It is well known that the second 0^+-state in ^{16}O at 6.06 MeV has a structure of an α-particle orbiting in an S-wave around a ^{12}C core [10,13,14]. Exciting this ^{12}C core to the Hoyle state we find the excitation energy 7.65 MeV + 6.06 MeV = 13.71 MeV. Of course, this close agreement may be a coincidence and more experimental evidences are needed.

The experimental determination of the corresponding form factor would be highly welcome and an eventual agreement with our calculated result, we think, a clear indication of the dilute α-particle structure of the corresponding 0^+-state in ^{16}O.

In conclusion we showed that the reproduction of the experimental inelastic form factor $0_1^+ \to 0_2^+$ of ^{12}C from our and Kamimura *et al.* wave functions is highly non-trivial due to the high sensitivity on the spatial extension of the Hoyle state. Together with the reproduction of other experimental data like for instance the transition probability [4,15], we believe that the almost ideal Bose condensate nature of the Hoyle state is now firmly established. We also made predictions for the inelastic form factor to the 2_2^+-state in ^{12}C which we interpreted in [7] as a quadrupole particle-hole excitation of the Hoyle state. A prediction of the form factor $0_1^+ \to \alpha$ condensate state in ^{16}O is also presented and it is argued that an experimental confirmation of this form factor undoubtedly would reveal the condensate character of the corresponding state.

This work was partially performed in the Research Project for the Study of Unstable Nuclei from Nuclear Cluster Aspects sponsored by the Institute of Physical and Chemical Research (RIKEN), and is supported by the Ministry of Education, Culture, Sports, Science and Technology (MEXT), Grant-in-Aid for JSPS Fellows (No. 03J05511) and by the Grant-in-Aid for the 21st Century COE "Center for Diversity and Universality in Physics" from the MEXT of Japan. This work was done as a part of the Japan-France Research Cooperative Program under CNRS/JSPS bilateral agreement.

References

1. F. Hoyle, Astrophys. J., Suppl. **1**, 121 (1954).
2. Y. Fukushima, M. Kamimura, *Proceedings of the International Conference on Nuclear Structure, Tokyo, 1977*, edited by T. Marumori (J. Phys. Soc. Jpn. Suppl. **44**, 225 (1978)); M. Kamimura, Nucl. Phys. A **351**, 456 (1981).
3. E. Uegaki, S. Okabe, Y. Abe, H. Tanaka, Prog. Theor. Phys. **57**, 1262 (1977); E. Uegaki, Y. Abe, S. Okabe, H. Tanaka, Prog. Theor. Phys. **59**, 1031 (1978); **62**, 1621 (1979).
4. A. Tohsaki, H. Horiuchi, P. Schuck, G. Röpke, Phys. Rev. Lett. **87**, 192501 (2001).
5. Y. Funaki, A. Tohsaki, H. Horiuchi, P. Schuck, G. Röpke, Phys. Rev. C **67**, 051306(R) (2003).
6. I. Sick, J.S. McCarthy, Nucl. Phys. A **150**, 631 (1970); A. Nakada, Y. Torizuka, Y. Horikawa, Phys. Rev. Lett. **27**, 745 (1971); **27**, 1102 (1971)(E); P. Strehl, Th.H. Schucan, Phys. Lett. B **27**, 641 (1968).

7. Y. Funaki, A. Tohsaki, H. Horiuchi, P. Schuck, G. Röpke, Eur. Phys. J. A **24**, 321 (2005).
8. Y. Funaki, H. Horiuchi, A. Tohsaki, Prog. Theor. Phys. **115**, 115 (2006).
9. M. Itoh *et al.*, *Proceedings of the 8th International Conference on Clustering Aspects of Nuclear Structure and Dynamics, Nara, Japan, 2003*, edited by K. Ikeda, I. Tanihata, H. Horiuchi (Nucl. Phys. A **738**, 268 (2004)).
10. For example, Y. Fujiwara, H. Horiuchi, K. Ikeda, M. Kamimura, K. Katō, Y. Suzuki, E. Uegaki, Prog. Theor. Phys. Suppl. **68**, 29 (1980); P. Descouvemont, D. Baye, Phys. Rev. C **36**, 54 (1987).
11. The $\boldsymbol{\beta}_1$ value corresponds to a position of the minimum in fig. 2 of [5]. This may indicate a strongly oblate minimum in contradiction with the spherical β-values employed here to vary the size of the Hoyle state. It should be recognized, however, in looking at fig. 2 of [5], that the minimum at $\boldsymbol{\beta} = (\beta_x = \beta_y, \beta_z) = (5.7\,\text{fm}, 1.3\,\text{fm})$ is extremely shallow and connected by an almost flat valley to the spherical point. The squared overlap of both wave functions at the minimum and spherical points on the valley is more than 90%. This is also underlined by the fact that the $\boldsymbol{\beta}_1$ position (cross) in fig. 4 only is very little off the continuous line.
12. T. Wakasa, private communication. See also `http://www.rcnp.osaka-u.ac.jp/annurep/2002/sec1/wakasa2.pdf`.
13. H. Horiuchi, K. Ikeda, Prog. Theor. Phys. **40**, 277 (1968).
14. K. Wildermuth, Y.C. Tang, *A Unified Theory of the Nucleus* (Vieweg, Braunschweig, Germany, 1977).
15. F. Ajzenberg-Selove, Nucl. Phys. A **248**, 1 (1975).